3.4 — Multivariate OLS Estimators

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Outline

The Multivariate OLS Estimators

The Multivariate OLS Estimators

- By analogy, we still focus on the **ordinary least squares (OLS) estimators** of the unknown population parameters $\beta_0, \beta_1, \beta_2, \cdots, \beta_k$ which solves:

$$\min_{\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \cdots, \hat{\beta}_k} \sum_{i=1}^{n} \Bigg[\underbrace{Y_i - \underbrace{(\hat{\beta}_0 + \hat{\beta}_1 X_{1i} + \hat{\beta}_2 X_{2i} + \cdots + \hat{\beta}_k X_{ki})}_{\hat{Y}_i}}_{u_i} \Bigg]^2$$

- Again, OLS estimators are chosen to **minimize** the **sum of squared errors (SSE)**
  - i.e. sum of squared distances between actual values of Y_i and predicted values $\hat{Y}_i$

The Multivariate OLS Estimators: FYI

Math FYI: in linear algebra terms, a regression model with n observations of k independent variables:

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{u}$$

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{\mathbf{Y}_{(n \times 1)}} = \underbrace{\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k,1} & x_{k,2} & \cdots & x_{k,n} \end{pmatrix}}_{\mathbf{X}_{(n \times k)}} \underbrace{\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}}_{\beta_{(k \times 1)}} + \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}}_{\mathbf{u}_{(n \times 1)}}$$

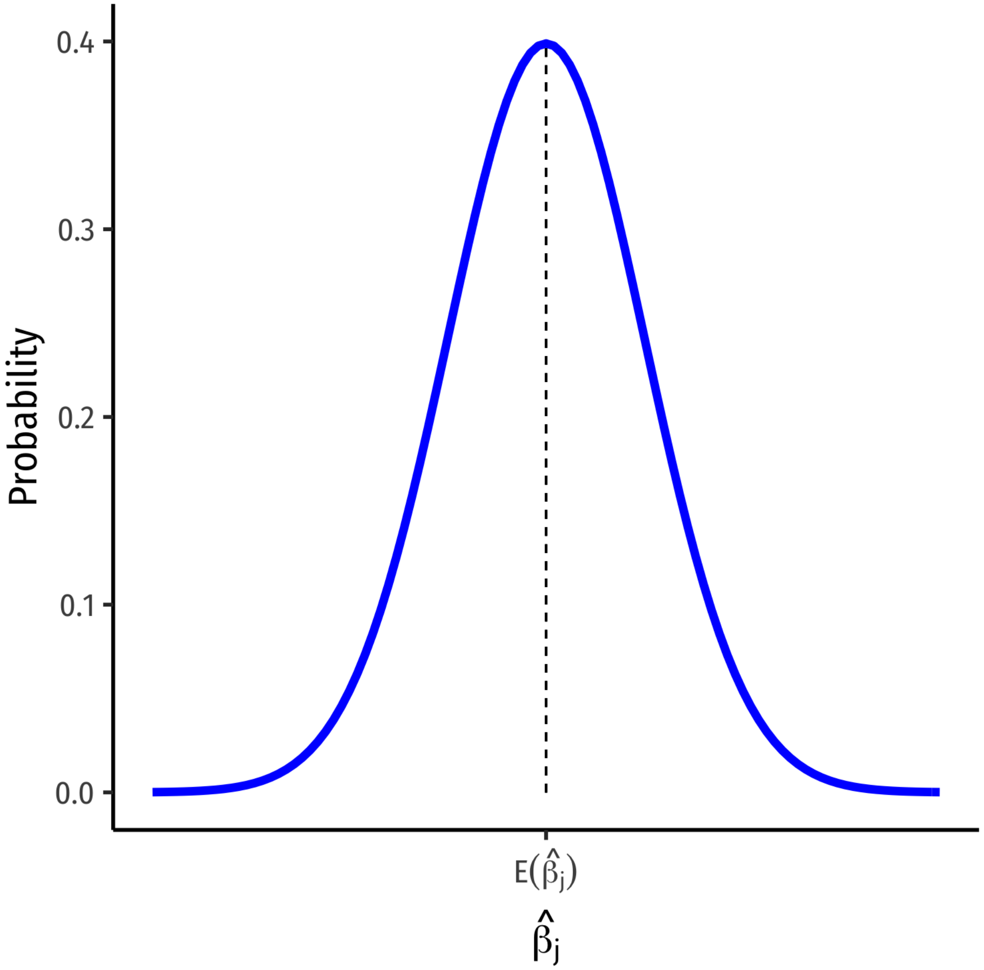
- The OLS estimator for β is $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ 😱
- Appreciate that I am saving you from such sorrow 🤖

The Sampling Distribution of $\hat{\beta}_j$

- For *any* individual β_j, it has a sampling distribution:

$$\hat{\beta}_j \sim N\left(E[\hat{\beta}_j],\ se(\hat{\beta}_j)\right)$$

- We want to know its sampling distribution's:
  - **Center**: $E[\hat{\beta}_j]$; what is the *expected value* of our estimator?
  - **Spread**: $se(\hat{\beta}_j)$; how *precise* or *uncertain* is our estimator?



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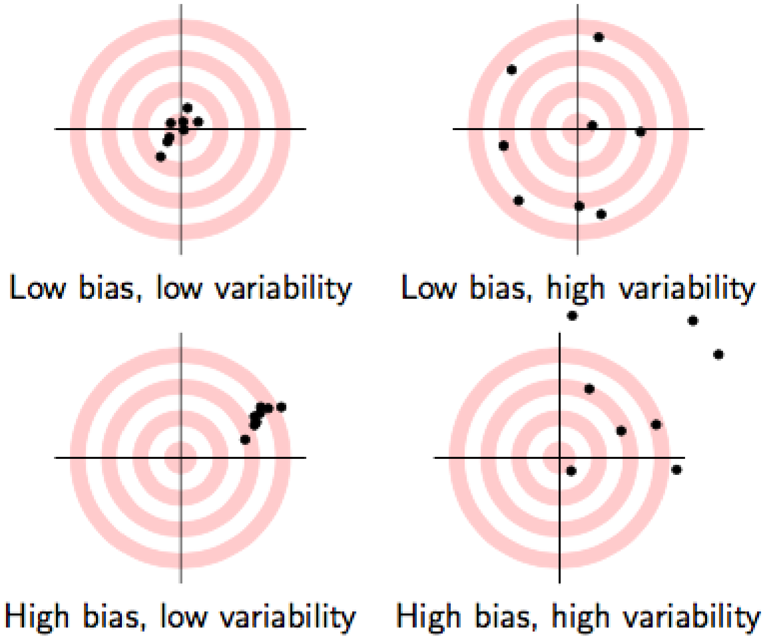


The Expected Value of $\hat{\beta}_j$: Bias

Exogeneity and Unbiasedness

- As before, $E[\hat{\beta}_j] = \beta_j$ when X_j is **exogenous** (i.e. $cor(X_j, u) = 0$)

- We know the true $E[\hat{\beta}_j] = \beta_j + \underbrace{cor(X_j, u)\frac{\sigma_u}{\sigma_{X_j}}}_{\text{O.V. Bias}}$

- If X_j is **endogenous** (i.e. $cor(X_j, u) \neq 0$), contains **omitted variable bias**

- We can now try to *quantify* the omitted variable bias

Measuring Omitted Variable Bias I

- Suppose the ***true* population model** of a relationship is:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i$$

- What happens when we run a regression and **omit** X_{2i}?

- Suppose we estimate the following **omitted regression** of just Y_i on X_{1i} (omitting X_{2i}):†

$$Y_i = \alpha_0 + \alpha_1 X_{1i} + \nu_i$$

† Note: I am using α's and ν_i only to denote these are different estimates than the **true** model β's and u_i

Measuring Omitted Variable Bias II

- **Key Question:** are X_{1i} and X_{2i} correlated?

- Run an **auxiliary regression** of X_{2i} on X_{1i} to see:[†]

$$X_{2i} = \delta_0 + \delta_1 X_{1i} + \tau_i$$

- If $\delta_1 = 0$, then X_{1i} and X_{2i} are *not* linearly related

- If $|\delta_1|$ is very big, then X_{1i} and X_{2i} are strongly linearly related

[†] Note: I am using δ's and τ to differentiate estimates for this model.

Measuring Omitted Variable Bias III

- Now substitute our **auxiliary regression** between X_{2i} and X_{1i} into the ***true* model**:
  - We know $X_{2i} = \delta_0 + \delta_1 X_{1i} + \tau_i$

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i$$

Measuring Omitted Variable Bias III

- Now substitute our **auxiliary regression** between X_{2i} and X_{1i} into the ***true* model**:
  - We know $X_{2i} = \delta_0 + \delta_1 X_{1i} + \tau_i$

$$
\begin{aligned}
Y_i &= \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i \\
Y_i &= \beta_0 + \beta_1 X_{1i} + \beta_2 \big(\delta_0 + \delta_1 X_{1i} + \tau_i\big) + u_i
\end{aligned}
$$

Measuring Omitted Variable Bias III

- Now substitute our **auxiliary regression** between X_{2i} and X_{1i} into the ***true* model**:
  - We know $X_{2i} = \delta_0 + \delta_1 X_{1i} + \tau_i$

$$
\begin{aligned}
Y_i &= \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i \\
Y_i &= \beta_0 + \beta_1 X_{1i} + \beta_2 \big(\delta_0 + \delta_1 X_{1i} + \tau_i\big) + u_i \\
Y_i &= (\beta_0 + \beta_2 \delta_0) + (\beta_1 + \beta_2 \delta_1) X_{1i} + (\beta_2 \tau_i + u_i)
\end{aligned}
$$

Measuring Omitted Variable Bias III

- Now substitute our **auxiliary regression** between X_{2i} and X_{1i} into the ***true* model**:
  - We know $X_{2i} = \delta_0 + \delta_1 X_{1i} + \tau_i$

$$
\begin{aligned}
Y_i &= \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i \\
Y_i &= \beta_0 + \beta_1 X_{1i} + \beta_2 \big(\delta_0 + \delta_1 X_{1i} + \tau_i\big) + u_i \\
Y_i &= \underbrace{(\beta_0 + \beta_2 \delta_0)}_{\alpha_0} + \underbrace{(\beta_1 + \beta_2 \delta_1)}_{\alpha_1} X_{1i} + \underbrace{(\beta_2 \tau_i + u_i)}_{\nu_i}
\end{aligned}
$$

- Now relabel each of the three terms as the OLS estimates (α's) and error (ν_i) from the **omitted regression**, so we again have:

$$Y_i = \alpha_0 + \alpha_1 X_{1i} + \nu_i$$

- Crucially, this means that our OLS estimate for X_{1i} in the **omitted regression** is:

$$\alpha_1 = \beta_1 + \beta_2 \delta_1$$

Measuring Omitted Variable Bias IV

$$\alpha_1 = \beta_1 + \beta_2 \delta_1$$

- The **Omitted Regression** OLS estimate for X_{1i}, (α_1) picks up *both*:

1) The true effect of X_1 on Y_i: (β_1)

2) The true effect of X_2 on Y_i: (β_2)

- As pulled through the relationship between X_1 and X_2: (δ_1)

- Recall our conditions for omitted variable bias from some variable Z_i:

1) $\mathbf{Z_i}$ **must be a determinant of** Y_i $\implies \beta_2 \neq 0$

2) $\mathbf{Z_i}$ **must be correlated with** X_i $\implies \delta_1 \neq 0$

- Otherwise, if Z_i does not fit these conditions, $\alpha_1 = \beta_1$ and the **omitted regression** is *unbiased*!

Measuring OVB in Our Class Size Example I

- The **"True" Regression** $(Y_i$ on X_{1i} and $X_{2i})$

$$\widehat{\text{Test Score}_i} = 686.03 - 1.10\ \text{STR}_i - 0.65\ \%\text{EL}_i$$

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
(Intercept)	686.0322487	7.41131248	92.565554	3.871501e-280
str	-1.1012959	0.38027832	-2.896026	3.978056e-03
el_pct	-0.6497768	0.03934255	-16.515879	1.657506e-47

3 rows

Measuring OVB in Our Class Size Example II

- The **"Omitted" Regression** (Y_i on just X_{1i})

$$\widehat{\text{Test Score}_i} = 698.93 - 2.28 \text{ STR}_i$$

term <chr>	estimate <dbl>	std.error <dbl>	statistic <dbl>	p.value <dbl>
(Intercept)	698.932952	9.4674914	73.824514	6.569925e-242
str	-2.279808	0.4798256	-4.751327	2.783307e-06

2 rows

Measuring OVB in Our Class Size Example III

- The **"Auxiliary" Regression** $(X_{2i}$ on $X_{1i})$

$$\widehat{\%\text{EL}_i} = -19.85 + 1.81 \text{ STR}_i$$

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
(Intercept)	-19.854055	9.1626044	-2.166857	0.0308099863
str	1.813719	0.4643735	3.905733	0.0001095165

2 rows

Measuring OVB in Our Class Size Example IV

"True" Regression

$$\widehat{\text{Test Score}_i} = 686.03 - 1.10\ \text{STR}_i - 0.65\ \%\text{EL}$$

"Omitted" Regression

$$\widehat{\text{Test Score}_i} = 698.93 - 2.28\ \text{STR}_i$$

"Auxiliary" Regression

$$\widehat{\%\text{EL}_i} = -19.85 + 1.81\ \text{STR}_i$$

- Omitted Regression α_1 on STR is -2.28

Measuring OVB in Our Class Size Example IV

"True" Regression

$$\widehat{\text{Test Score}_i} = 686.03 - 1.10\ \text{STR}_i - 0.65\ \%\text{EL}$$

"Omitted" Regression

$$\widehat{\text{Test Score}_i} = 698.93 - 2.28\ \text{STR}_i$$

"Auxiliary" Regression

$$\widehat{\%\text{EL}_i} = -19.85 + 1.81\ \text{STR}_i$$

- Omitted Regression α_1 on STR is -2.28

$$\alpha_1 = \beta_1 + \beta_2\delta_1$$

- The true effect of STR on Test Score: -1.10

Measuring OVB in Our Class Size Example IV

"True" Regression

$$\widehat{\text{Test Score}_i} = 686.03 - 1.10\ \text{STR}_i - 0.65\ \%\text{EL}$$

"Omitted" Regression

$$\widehat{\text{Test Score}_i} = 698.93 - 2.28\ \text{STR}_i$$

"Auxiliary" Regression

$$\widehat{\%\text{EL}_i} = -19.85 + 1.81\ \text{STR}_i$$

- Omitted Regression α_1 on STR is -2.28

$$\alpha_1 = \beta_1 + \beta_2 \delta_1$$

- The true effect of STR on Test Score: -1.10
- The true effect of %EL on Test Score: -0.65

Measuring OVB in Our Class Size Example IV

"True" Regression

$$\widehat{\text{Test Score}_i} = 686.03 - 1.10\ \text{STR}_i - 0.65\ \%\text{EL}$$

"Omitted" Regression

$$\widehat{\text{Test Score}_i} = 698.93 - 2.28\ \text{STR}_i$$

"Auxiliary" Regression

$$\widehat{\%\text{EL}_i} = -19.85 + 1.81\ \text{STR}_i$$

- Omitted Regression α_1 on STR is -2.28

$$\alpha_1 = \beta_1 + \beta_2 \delta_1$$

- The true effect of STR on Test Score: -1.10
- The true effect of %EL on Test Score: -0.65
- The relationship between STR and %EL: 1.81

Measuring OVB in Our Class Size Example IV

"True" Regression

$$\widehat{\text{Test Score}_i} = 686.03 - 1.10\ \text{STR}_i - 0.65\ \%\text{EL}$$

"Omitted" Regression

$$\widehat{\text{Test Score}_i} = 698.93 - 2.28\ \text{STR}_i$$

"Auxiliary" Regression

$$\widehat{\%\text{EL}_i} = -19.85 + 1.81\ \text{STR}_i$$

- Omitted Regression α_1 on STR is -2.28

$$\alpha_1 = \beta_1 + \beta_2\delta_1$$

- The true effect of STR on Test Score: -1.10
- The true effect of %EL on Test Score: -0.65
- The relationship between STR and %EL: 1.81
- So, for the **omitted regression**:

$$-2.28 = -1.10 + (-0.65)(1.81)$$

Measuring OVB in Our Class Size Example IV

"True" Regression

$$\widehat{\text{Test Score}_i} = 686.03 - 1.10\ \text{STR}_i - 0.65\ \%\text{EL}$$

"Omitted" Regression

$$\widehat{\text{Test Score}_i} = 698.93 - 2.28\ \text{STR}_i$$

"Auxiliary" Regression

$$\widehat{\%\text{EL}_i} = -19.85 + 1.81\ \text{STR}_i$$

- Omitted Regression α_1 on STR is -2.28

$$\alpha_1 = \beta_1 + \beta_2\delta_1$$

- The true effect of STR on Test Score: -1.10
- The true effect of %EL on Test Score: -0.65
- The relationship between STR and %EL: 1.81
- So, for the **omitted regression**:

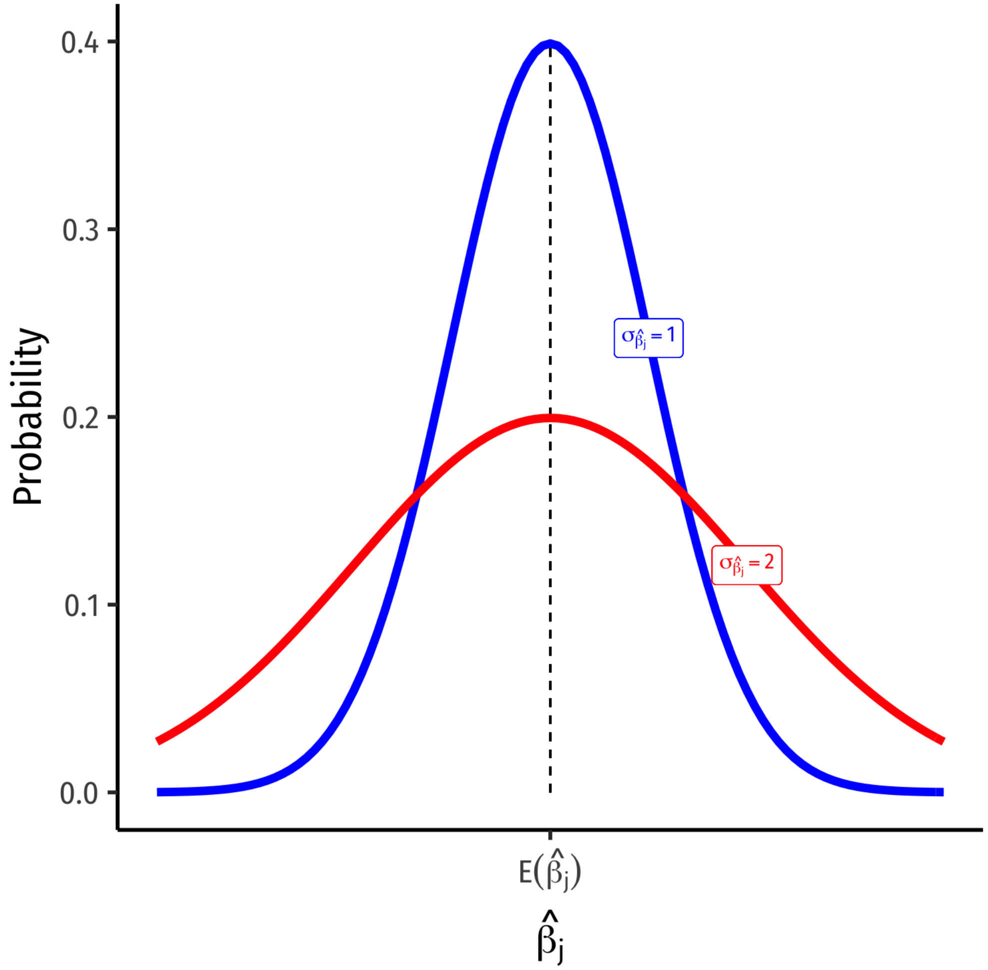
$$-2.28 = -1.10 + \underbrace{(-0.65)(1.81)}_{O.V.Bias = \mathbf{-1.18}}$$

Precision of $\hat{\beta}_j$

Precision of $\hat{\beta}_j$ I

- $\sigma_{\hat{\beta}_j}$; how **precise** are our estimates?

- **Variance $\sigma^2_{\hat{\beta}_j}$** or **standard error $\sigma_{\hat{\beta}_j}$**



Precision of $\hat{\beta_j}$ II

$$var(\hat{\beta_j}) = \underbrace{\frac{1}{1-R_j^2}}_{VIF} \times \frac{(SER)^2}{n \times var(X)}$$

$$se(\hat{\beta_j}) = \sqrt{var(\hat{\beta_1})}$$

- Variation in $\hat{\beta_j}$ is affected by **four** things now[†]:

1. **Goodness of fit of the model (SER)**
   - Larger SER → larger $var(\hat{\beta_j})$
2. **Sample size, *n***
   - Larger n → smaller $var(\hat{\beta_j})$
3. **Variance of X**
   - Larger $var(X)$ → smaller $var(\hat{\beta_j})$
4. **Variance Inflation Factor** $\frac{1}{(1-R_j^2)}$
   - Larger VIF, larger $var(\hat{\beta_j})$
   - **This is the only new effect**

[†] See Class 2.5 for a reminder of variation with just one X variable.

VIF and Multicollinearity I

- Two *independent* (X) variables are **multicollinear**:

$$cor(X_j, X_l) \neq 0 \quad \forall j \neq l$$

- **Multicollinearity between X variables does *not bias* OLS estimates**
  - Remember, we pulled another variable out of u into the regression
  - If it were omitted, then it *would* cause omitted variable bias!
- **Multicollinearity does *increase the variance* of each estimate** by

$$VIF = \frac{1}{(1 - R_j^2)}$$

VIF and Multicollinearity II

$$VIF = \frac{1}{(1 - R_j^2)}$$

- R_j^2 is the R^2 from an **auxiliary regression** of X_j on all other regressors (X's)

Example: Suppose we have a regression with three regressors ($k = 3$):

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i}$$

- There will be three different R_j^2's, one for each regressor:

$$\begin{aligned} R_1^2 \text{ for } X_{1i} &= \gamma + \gamma X_{2i} + \gamma X_{3i} \\ R_2^2 \text{ for } X_{2i} &= \zeta_0 + \zeta_1 X_{1i} + \zeta_2 X_{3i} \\ R_3^2 \text{ for } X_{3i} &= \eta_0 + \eta_1 X_{1i} + \eta_2 X_{2i} \end{aligned}$$

VIF and Multicollinearity III

$$VIF = \frac{1}{(1 - R_j^2)}$$

- R_j^2 is the R^2 from an **auxiliary regression** of X_j on all other regressors (X's)

- The R_j^2 tells us **how much *other* regressors explain regressor X_j**

- **Key Takeaway**: If other X variables explain X_j well (high R_J^2), it will be harder to tell how *cleanly* $X_j \rightarrow Y_i$, and so $var(\hat{\beta}_j)$ will be higher

VIF and Multicollinearity IV

- Common to calculate the **Variance Inflation Factor (VIF)** for each regressor:

$$VIF = \frac{1}{(1 - R_j^2)}$$

- VIF quantifies the factor (scalar) by which $var(\hat{\beta}_j)$ increases because of multicollinearity
  - e.g. VIF of 2, 3, etc. $\implies$ variance increases by 2x, 3x, etc.

- Baseline: $R_j^2 = 0 \implies$ *no* multicollinearity $\implies VIF = 1$ (no inflation)

- Larger $R_j^2 \implies$ larger VIF

  - Rule of thumb: $VIF > 10$ is problematic

VIF and Multicollinearity V

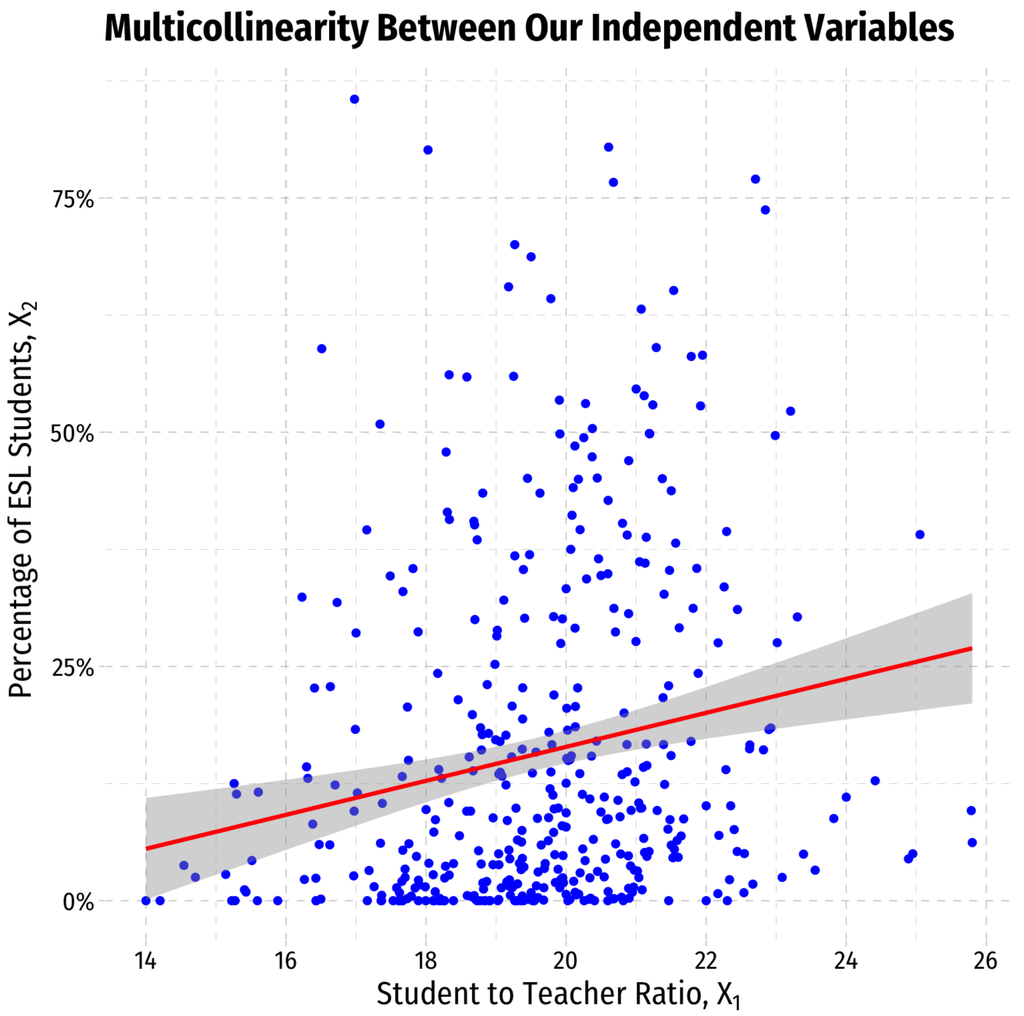
```r
# scatterplot of X2 on X1

ggplot(data=CASchool, aes(x=str,y=el_pct))+
  geom_point(color="blue")+
  geom_smooth(method="lm", color="red")+
  scale_y_continuous(labels=function(x){paste0(x,"%")})+
  labs(x = expression(paste("Student to Teacher Ratio, ", X[1])),
       y = expression(paste("Percentage of ESL Students, ", X[2])),
       title = "Multicollinearity Between Our Independent Variables")+
    ggthemes::theme_pander(base_family = "Fira Sans Condensed",
           base_size=16)
```

```r
# Make a correlation table
CASchool %>%
  select(testscr, str, el_pct) %>%
  cor()
```

```
##             testscr        str     el_pct
## testscr  1.0000000 -0.2263628 -0.6441237
## str     -0.2263628  1.0000000  0.1876424
## el_pct  -0.6441237  0.1876424  1.0000000
```

- Cor(STR, %EL) = -0.644



VIF and Multicollinearity in R I

```r
# our multivariate regression
elreg <- lm(testscr ~ str + el_pct,
            data = CASchool)

# use the "car" package for VIF function
library("car")

# syntax: vif(lm.object)
vif(elreg)
```

```
##      str   el_pct 
## 1.036495 1.036495
```

- $var(\hat{\beta}_1)$ on `str` increases by **1.036** times (3.6%) due to multicollinearity with `el_pct`
- $var(\hat{\beta}_2)$ on `el_pct` increases by **1.036** times (3.6%) due to multicollinearity with `str`

VIF and Multicollinearity in R II

- Let's calculate VIF manually to see where it comes from:

```
# run auxiliary regression of x2 on x1

auxreg <- lm(el_pct ~ str,
             data = CASchool)

# use broom package's tidy() command (cleaner)

library(broom) # load broom

tidy(auxreg) # look at reg output
```

term <chr>	estimate <dbl>	std.error <dbl>	statistic <dbl>	p.value <dbl>
(Intercept)	-19.854055	9.1626044	-2.166857	0.0308099863
str	1.813719	0.4643735	3.905733	0.0001095165

2 rows

VIF and Multicollinearity in R III

```
glance(auxreg) # look at aux reg stats for R^2
```

r.squared	adj.r.squared	sigma	statistic	p.value	df	logLik	AIC	BIC
<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
0.03520966	0.03290155	17.98259	15.25475	0.0001095165	1	-1808.502	3623.003	3635.124

1 row | 1-9 of 12 columns

```
# extract our R-squared from aux regression (R_j^2)

aux_r_sq <- glance(auxreg) %>%

  select(r.squared)
```

```
aux_r_sq # look at it
```

r.squared
<dbl>
0.03520966

VIF and Multicollinearity in R IV

```r
# calculate VIF manually

our_vif <- 1 / (1 - aux_r_sq) # VIF formula

our_vif
```

r.squared
<dbl>
1.036495

1 row

- Again, multicollinearity between the two X variables inflates the variance on each by 1.036 times

VIF and Multicollinearity: Another Example I

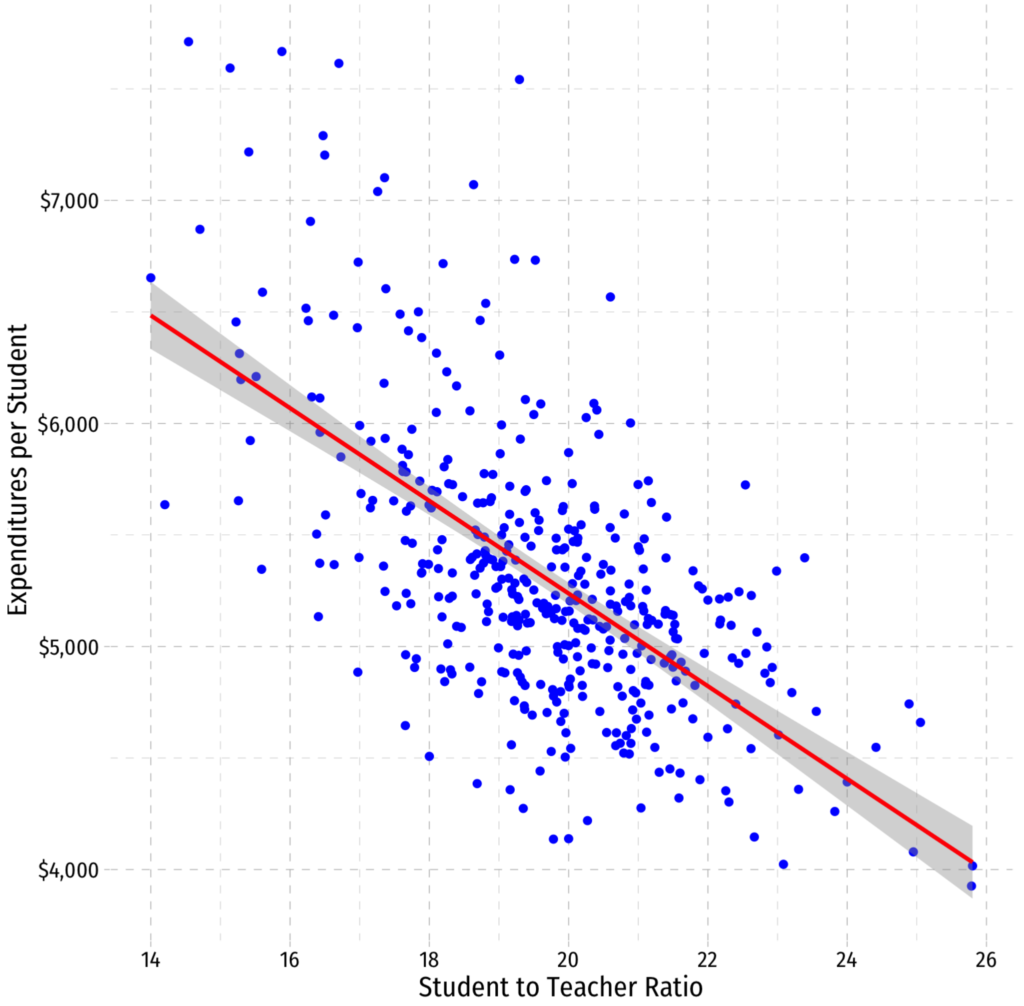
Example: What about district expenditures per student?

```
CASchool %>%
  select(testscr, str, expn_stu) %>%
  cor()
```

```
##             testscr        str   expn_stu
## testscr   1.0000000 -0.2263628  0.1912728
## str      -0.2263628  1.0000000 -0.6199821
## expn_stu  0.1912728 -0.6199821  1.0000000
```

VIF and Multicollinearity: Another Example II

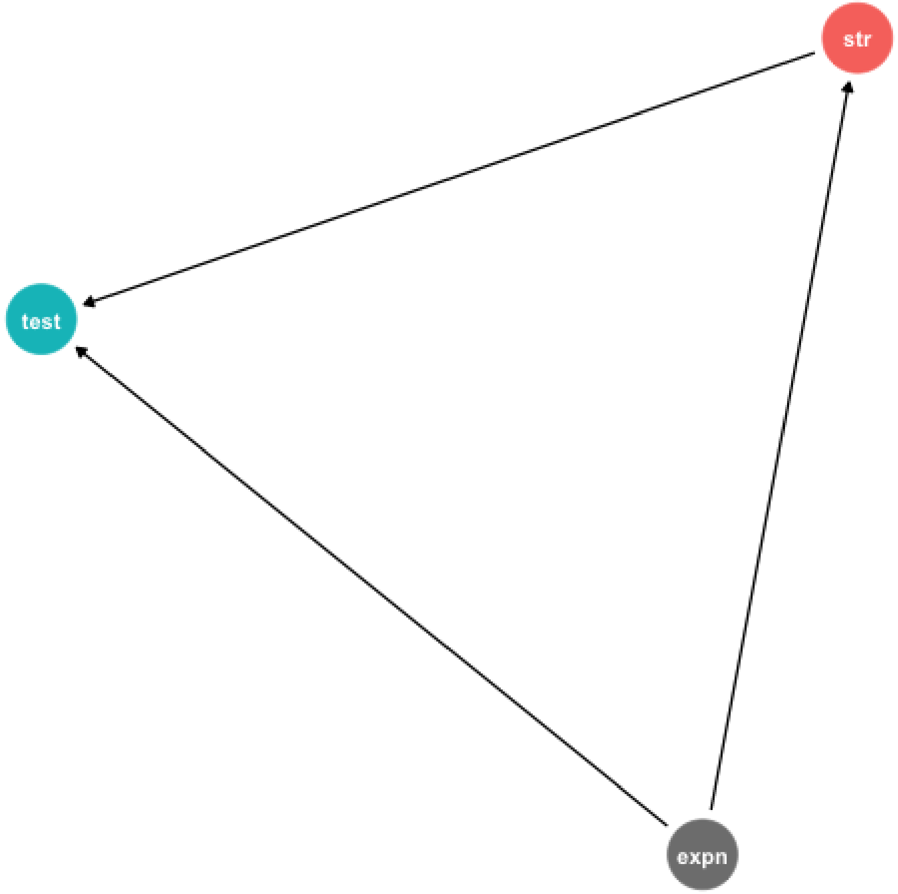
```
ggplot(data=CASchool, aes(x=str,y=expn_stu))+
  geom_point(color="blue")+
  geom_smooth(method="lm", color="red")+
  scale_y_continuous(labels = scales::dollar)+
  labs(x = "Student to Teacher Ratio",
       y = "Expenditures per Student")+
  ggthemes::theme_pander(base_family = "Fira Sans (
           base_size=14)
```



VIF and Multicollinearity: Another Example III

1. $cor(\text{Test score, expn}) \neq 0$

2. $cor(\text{STR, expn}) \neq 0$

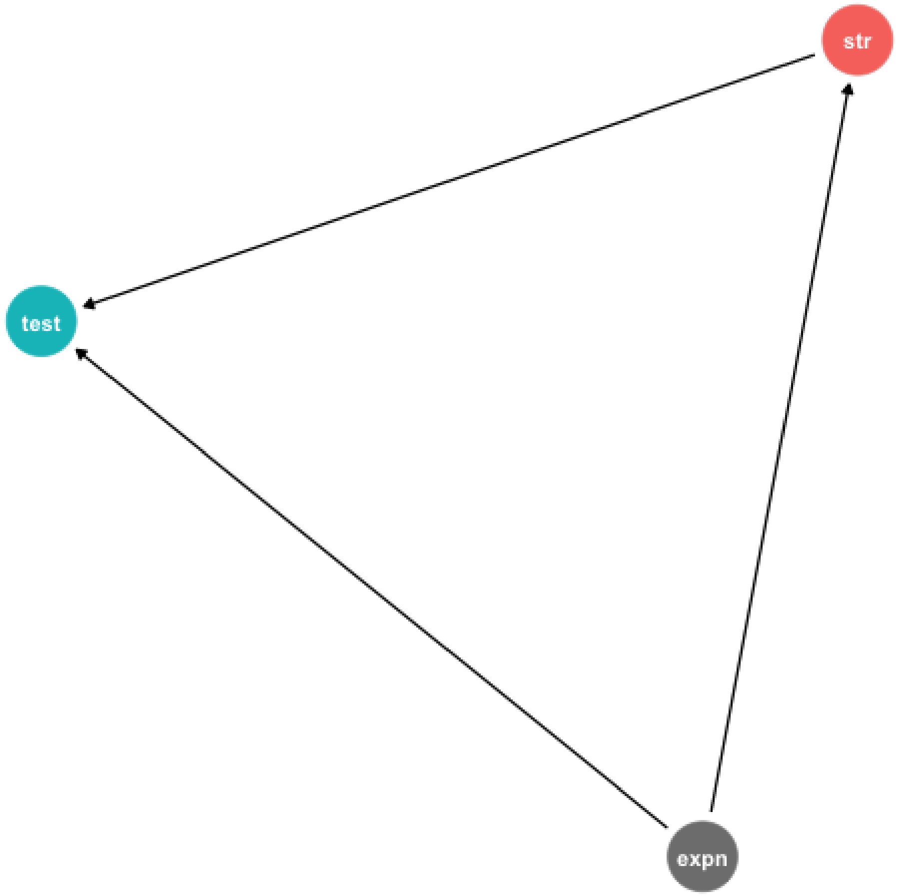


VIF and Multicollinearity: Another Example III

1. $cor(\text{Test score, expn}) \neq 0$

2. $cor(\text{STR, expn}) \neq 0$

- Omitting $expn$ will **bias** $\hat{\beta}_1$ on STR



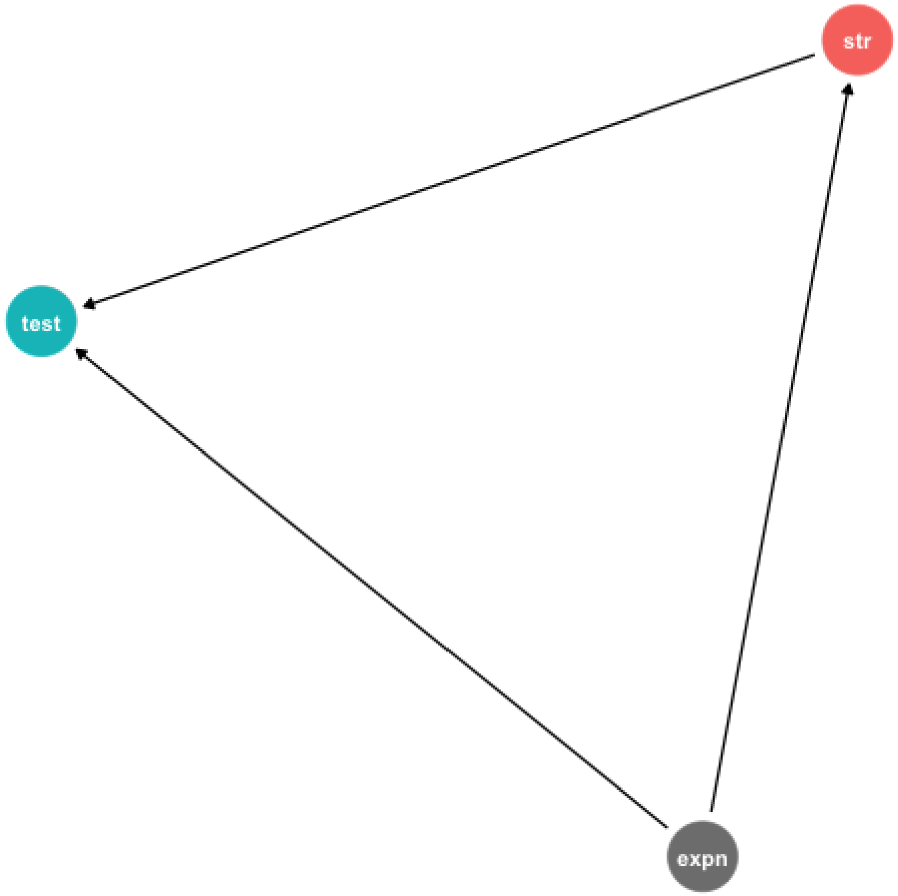
VIF and Multicollinearity: Another Example III

1. $cor(\text{Test score, expn}) \neq 0$

2. $cor(\text{STR, expn}) \neq 0$

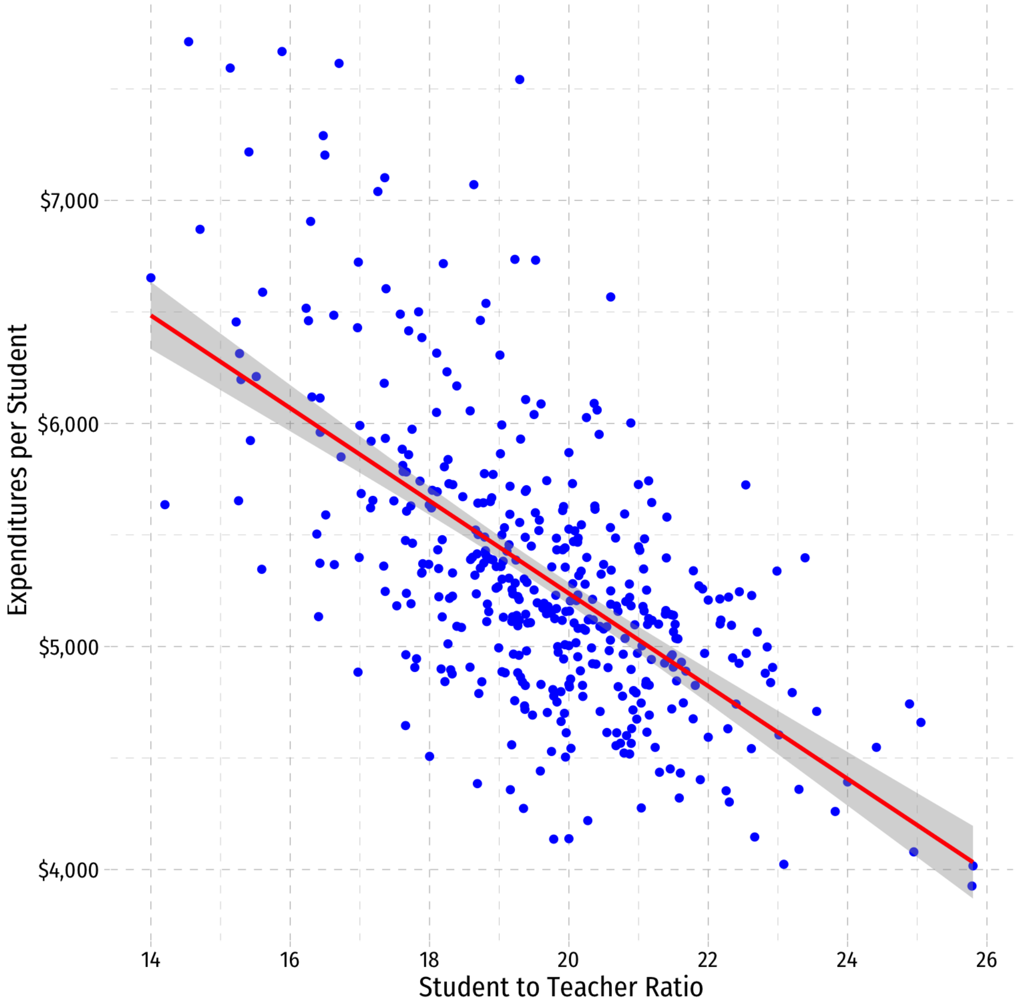
- Omitting $expn$ will **bias** $\hat{\beta}_1$ on STR

- *Including* $expn$ will *not* bias $\hat{\beta}_1$ on STR, but *will* make it less precise (higher variance)



VIF and Multicollinearity: Another Example III

- Data tells us little about the effect of a change in STR holding $expn$ constant
  - Hard to know what happens to test scores when high STR AND high $expn$ and vice versa (*they rarely happen simultaneously*)!



VIF and Multicollinearity: Another Example IV

```
expreg <- lm(testscr ~ str + expn_stu,
             data = CASchool)
expreg %>% tidy()
```

term	estimate	std.error	statistic
<chr>	<dbl>	<dbl>	<dbl>
(Intercept)	675.577173851	19.562221636	34.534788
str	-1.763215599	0.610913641	-2.886195
expn_stu	0.002486571	0.001823105	1.363921

3 rows | 1-4 of 5 columns

```
expreg %>%
  vif()
```

```
##      str expn_stu
## 1.624373 1.624373
```

- Including `expn_stu` increases variance of $\hat{\beta}_1$ and $\hat{\beta}_2$ by 1.62x (62%)

Multicollinearity Increases Variance

```r
library(huxtable)
huxreg("Model 1" = school_reg,
       "Model 2" = expreg,
       coefs = c("Intercept" = "(Intercept)",
                 "Class Size" = "str",
                 "Expenditures per Student" = "expn_stu"),
       statistics = c("N" = "nobs",
                      "R-Squared" = "r.squared",
                      "SER" = "sigma"),
       number_format = 2)
```

- We can see $SE(\hat{\beta}_1)$ on `str` increases from 0.48 to 0.61 when we add `expn_stu`

	Model 1	Model 2
Intercept	698.93 ***	675.58 ***
	(9.47)	(19.56)
Class Size	-2.28 ***	-1.76 **
	(0.48)	(0.61)
Expenditures per Student		0.00
		(0.00)
N	420	420
R-Squared	0.05	0.06
SER	18.58	18.56

*** p < 0.001; ** p < 0.01; * p < 0.05.

Perfect Multicollinearity

- *Perfect* **multicollinearity** is when a regressor is an exact linear function of (an)other regressor(s)

$$\widehat{Sales} = \hat{\beta}_0 + \hat{\beta}_1 \text{Temperature (C)} + \hat{\beta}_2 \text{Temperature (F)}$$

$$\text{Temperature (F)} = 32 + 1.8 * \text{Temperature (C)}$$

- $cor(\text{temperature (F), temperature (C)}) = 1$
- $R^2_j = 1$ is implying $VIF = \frac{1}{1-1}$ and $var(\hat{\beta}_j) = 0$!
- **This is fatal for a regression**
  - A logical impossiblity, **always caused by human error**

Perfect Multicollinearity: Example

Example:

$$\widehat{TestScore_i} = \hat{\beta}_0 + \hat{\beta}_1 STR_i + \hat{\beta}_2 \%EL + \hat{\beta}_3 \%EF$$

- $\%EL$: the percentage of students learning English
- $\%EF$: the percentage of students fluent in English
- $\%EF = 100 - \%EL$
- $|cor(\%EF, \%EL)| = 1$

Perfect Multicollinearity Example II

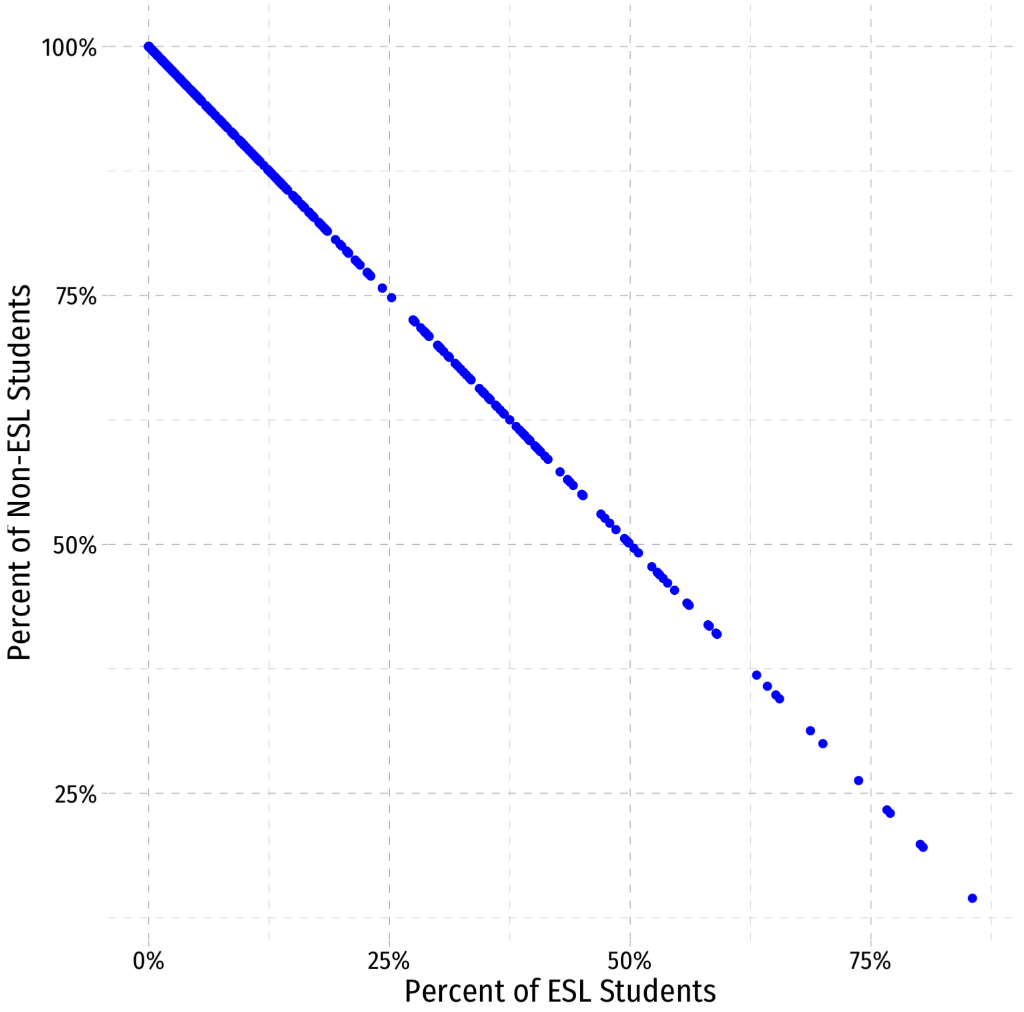
```r
# generate %EF variable from %EL
CASchool_ex <- CASchool %>%
  mutate(ef_pct = 100 - el_pct)

# get correlation between %EL and %EF
CASchool_ex %>%
  summarize(cor = cor(ef_pct, el_pct))
```

```
## # A tibble: 1 × 1
##     cor
##   <dbl>
## 1    -1
```

Perfect Multicollinearity Example III

```
ggplot(data = CASchool_ex)+
  aes(x = el_pct,
      y = ef_pct)+
  geom_point(color = "blue")+
  scale_x_continuous(labels = scales::percent_forma
  scale_y_continuous(labels = scales::percent_forma
  labs(x = "Percent of ESL Students",
       y = "Percent of Non-ESL Students")+
  ggthemes::theme_pander(base_family = "Fira Sans C
           base_size=16)
```



Perfect Multicollinearity Example IV

```r
mcreg <- lm(testscr ~ str + el_pct + ef_pct,
            data = CASchool_ex)
summary(mcreg)
```

```
##
## Call:
## lm(formula = testscr ~ str + el_pct + ef_pct, data = CASchool_ex)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -48.845 -10.240  -0.308   9.815  43.461
##
## Coefficients: (1 not defined because of singularities)
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 686.03225    7.41131  92.566  < 2e-16 ***
## str          -1.10130    0.38028  -2.896  0.00398 **
## el_pct       -0.64978    0.03934 -16.516  < 2e-16 ***
## ef_pct             NA         NA      NA       NA
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 14.46 on 417 degrees of freedom
## Multiple R-squared:  0.4264,    Adjusted R-squared:  0.4237
## F-statistic:   155 on 2 and 417 DF,  p-value: < 2.2e-16
```

```r
mcreg %>% tidy()
```

```
## # A tibble: 4 × 5
##   term        estimate std.error statistic   p.value
##   <chr>          <dbl>     <dbl>     <dbl>     <dbl>
## 1 (Intercept)  686.        7.41       92.6  3.87e-280
## 2 str           -1.10      0.380      -2.90 3.98e-  3
## 3 el_pct        -0.650     0.0393    -16.5  1.66e- 47
## 4 ef_pct        NA        NA          NA   NA
```

- Note `R` *drops* one of the multicollinear regressors (`ef_pct`) if you include both 🤡

A Summary of Multivariate OLS Estimator Properties

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- $\hat{\beta}_j$ on X_j is biased only if there is an omitted variable (Z) such that:

  1. $cor(Y, Z) \neq 0$
  2. $cor(X_j, Z) \neq 0$

  - If Z is *included* and X_j is collinear with Z, this does *not* cause a bias

- $var[\hat{\beta}_j]$ and $se[\hat{\beta}_j]$ measure precision (or uncertainty) of estimate:

$$var[\hat{\beta}_j] = \frac{1}{(1 - R^2_j)} * \frac{SER^2}{n \times var[X_j]}$$

- VIF from multicollinearity: $\frac{1}{(1-R^2_j)}$
  - R^2_j for auxiliary regression of X_j on all other X's
  - mutlicollinearity does not bias $\hat{\beta}_j$ but raises its variance
  - *perfect* multicollinearity if X's are linear function of others

Updated Measures of Fit

(Updated) Measures of Fit

- Again, how well does a linear model fit the data?

- How much variation in Y_i is "explained" by variation in the model $(\hat{Y_i})$?

$$
\begin{aligned}
Y_i &= \hat{Y}_i + \hat{u}_i \\
\hat{u}_i &= Y_i - \hat{Y}_i
\end{aligned}
$$

(Updated) Measures of Fit: SER

- Again, the **Standard errror of the regression (SER)** estimates the standard error of u

$$SER = \frac{SSE}{n - \mathbf{k} - 1}$$

- A measure of the spread of the observations around the regression line (in units of Y), the average "size" of the residual

- **Only new change:** divided by $n - k - 1$ due to use of $k + 1$ degrees of freedom to first estimate β_0 and then all of the other β's for the k number of regressors†

† Again, because your textbook defines *k* as including the constant, the denominator would be *n-k* instead of *n-k-1*.

(Updated) Measures of Fit: R^2

$$\begin{aligned} R^2 &= \frac{ESS}{TSS} \\ &= 1 - \frac{SSE}{TSS} \\ &= (r_{X,Y})^2 \end{aligned}$$

- Again, R^2 is fraction of total variation in Y_i ("total sum of squares") that is explained by variation in predicted values ($\hat{Y}_i$, i.e. our model ("explained sum of squares")

$$\frac{var(\hat{Y})}{var(Y)}$$

Visualizing R^2

- **Total Variation in Y**: Areas A + D + E + G

$$TSS = \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

- **Variation in Y explained by X1 and X2**: Areas D + E + G

$$ESS = \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2$$

- **Unexplained variation in Y**: **Area A**

$$SSE = \sum_{i=1}^{n} (\hat{u}_i)^2$$

Compare with one X variable

$$R^2 = \frac{ESS}{TSS} = \frac{D + E + G}{A + D + E + G}$$

Visualizing R^2

```r
# make a function to calc. sum of sq. devs
sum_sq <- function(x){sum((x - mean(x))^2)}

# find total sum of squares
TSS <- elreg %>%
  augment() %>%
  summarize(TSS = sum_sq(testscr))

# find explained sum of squares
ESS <- elreg %>%
  augment() %>%
  summarize(TSS = sum_sq(.fitted))

# look at them and divide to get R^2
tribble(
  ~ESS, ~TSS, ~R_sq,
  ESS, TSS, ESS/TSS
  ) %>%
  knitr::kable()
```

ESS	TSS	R_sq
64864.3	152109.6	0.4264314

$$R^2 = \frac{ESS}{TSS} = \frac{D+E+G}{\color{red}{A}+D+E+G} = 0.426$$

(Updated) Measures of Fit: Adjusted $\bar{R}^2$

- Problem: R^2 **mechanically** increases *every* time a new variable is added (it reduces SSE!)
  - Think in the diagram: more area of Y covered by more X variables!
- This does **not** mean adding a variable *improves the fit of the model* per se, R^2 gets **inflated**
- We correct for this effect with the **adjusted** $\bar{R}^2$ which penalizes adding new variables:

$$\bar{R}^2 = 1 - \underbrace{\frac{n-1}{n-k-1}}_{penalty} \times \frac{SSE}{TSS}$$

- In the end, recall R^2 **was never that useful**†, so don't worry about the formula
  - Large sample sizes (n) make R^2 and $\bar{R}^2$ very close

† ...for measuring causal effects (our goal). It *is* useful if you care about prediction instead!

In R (base)

```
##
## Call:
## lm(formula = testscr ~ str + el_pct, data = CASchool)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -48.845 -10.240  -0.308   9.815  43.461
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 686.03225    7.41131  92.566  < 2e-16 ***
## str          -1.10130    0.38028  -2.896  0.00398 **
## el_pct       -0.64978    0.03934 -16.516  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 14.46 on 417 degrees of freedom
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```

- Base R^2 (`R` calls it "`Multiple R-squared`") went up
- `Adjusted R-squared` went down

In R (broom)

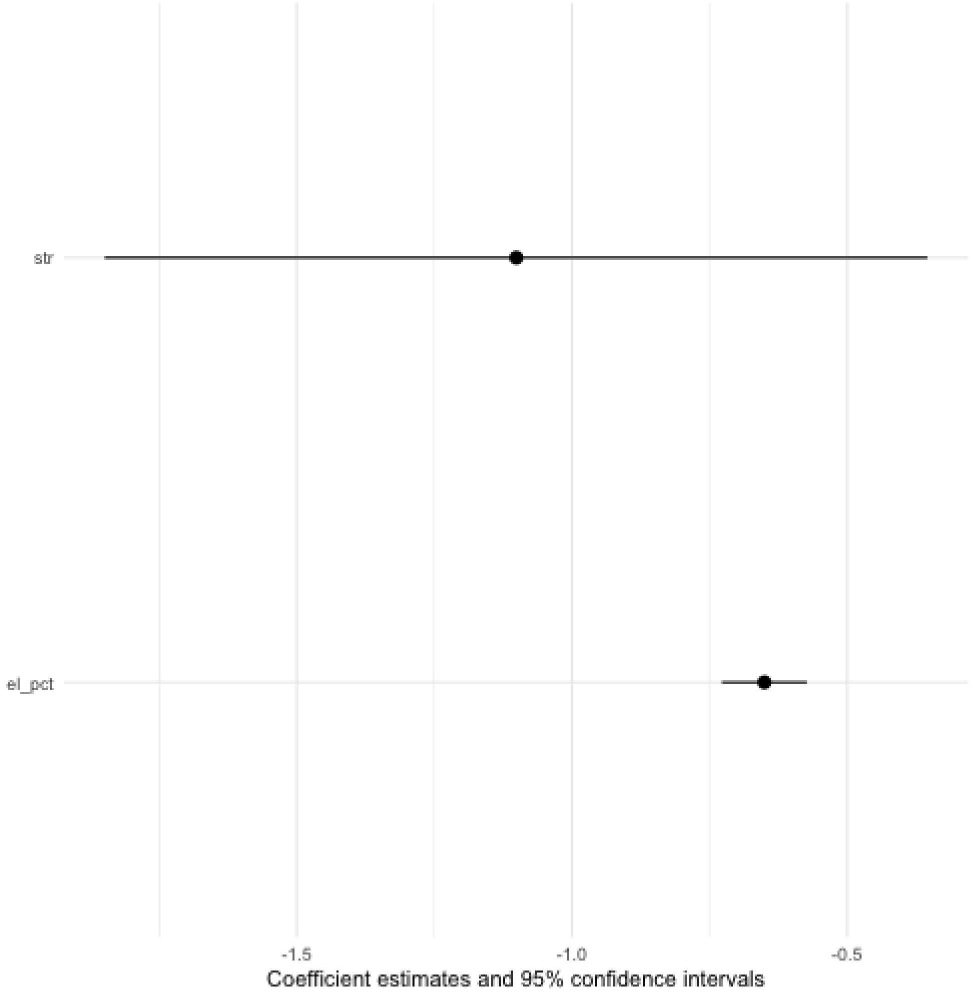
```
elreg %>%
  glance()
```

```
## # A tibble: 1 × 12
##   r.squared adj.r.squared sigma statistic  p.value    df logLik   AIC   BIC
##       <dbl>         <dbl> <dbl>     <dbl>    <dbl> <dbl>  <dbl> <dbl> <dbl>
## 1     0.426         0.424  14.5      155. 4.62e-51     2 -1717. 3441. 3457.
## # … with 3 more variables: deviance <dbl>, df.residual <int>, nobs <int>
```

Coefficient Plots

- The `modelsummary` package has a great command `modelplot()` for quickly making coefficient plots
- Learn more

```
library(modelsummary)
modelplot(elreg,  # our regression object
          coef_omit = 'Intercept') # don't show in
```



Modelsummary

- The `modelsummary` package also is a good alternative to `huxtable` for making regression tables (that's growing on me):
  - [Learn more]()

```r
modelsummary(models = list("Base Model" = school_reg,
                           "Multivariate Model" = elreg),
             fmt = 2, # round to 2 decimals
             output = "html",
             coef_rename = c("(Intercept)" = "Constant",
                             "str" = "STR",
                             "el_pct" = "% ESL Students"),
             gof_map = list(
               list("raw" = "nobs", "clean" = "N", "fmt" = 0),
               list("raw" = "r.squared", "clean" = "R<sup>2</sup>", "fmt" =
               list("raw" = "adj.r.squared", "clean" = "Adj. R<sup>2</sup>",
               list("raw" = "sigma", "clean" = "SER", "fmt" = 2)
             ),
             escape = FALSE,
             stars = TRUE
)
```

	Base Model	Multivariate Model
Constant	698.93***	686.03***
	(9.47)	(7.41)
STR	-2.28***	-1.10**
	(0.48)	(0.38)
% ESL Students		-0.65***
		(0.04)
N	420	420
R^2	0.05	0.43
Adj. R^2	0.05	0.42
SER	18.58	14.46

+ p < 0.1, * p < 0.05, ** p < 0.01, *** p < 0.001